Sketch of Solutions of Homework 9

#1 $C_n(X) \approx C_n(A) \oplus C_n(X,A)$ as abelian groups, not as chain complexes
$\therefore$ The isomorphism does not induce a homology isomorphism.
Example (E^n, S^{n-1})

#2 $\chi(X \times Y) = \sum_n (-1)^n \alpha_n(X \times Y) = \sum_n \sum_{i+j=n} (-1)^i \alpha_i(X) (-1)^j \alpha_j(Y)$

$= \left(\sum_r (-1)^r \alpha_r(X)\right)\left(\sum_s (-1)^s \alpha_s(Y)\right) = \chi(X)\chi(Y).$

#4 $\deg f = n$ for some n. $\therefore f \simeq p_n$ where $p_n z = z^n$. $p_n(1) = 1$.

#5 The open cells of $S^p \times S^q$ are

$\underbrace{e_0 \times e_0,\ e^p \times e_0,\ e_0 \times e^q}_{\text{This is } S^p \vee S^q = (p+q-1)\text{-skeleton}},\ e^p \times e^q$

the boundary of $e^p \times e^q \subseteq (p+q-1)$-skeleton
$\therefore$ $S^p \times S^q / S^p \vee S^q \approx$ closed $p+q$-cell / its boundary

#6 Let $f : \mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ and $q : S^{2n} \to \mathbb{R}P^{2n}$ the projection
Since q is a covering map and $\pi(S^{2n}) = 0$, f lifts to $\tilde{f} : S^{2n} \to S^{2n}$, $q\tilde{f} = fq$. Let $x_0 \in S^{2n}$ with $\tilde{f}(x_0) = x_0$ or $-x_0$
$\therefore q\tilde{f}(x_0) = [x_0]$ But $f[x_0] = fq(x_0) = q\tilde{f}(x_0) = [x_0]$.
$\therefore [x_0]$ is fixed point. Next let $T : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a LT without eigenvalues. $T : \mathbb{R}^{2n} - 0 \to \mathbb{R}^{2n} - 0$ (otherwise 0 is an eigenvalue)
Let $x, y \in \mathbb{R}^{2n} - 0$ and $x \sim y$ $\therefore \exists \lambda \neq 0$ $y = \lambda x$ $\therefore Ty = \lambda Tx$ so $Tx \sim Ty$. $\therefore T$ induces $\tilde{T} : \mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$. If $x_0 = [x]$ is a fixed point for $\tilde{T}$, $\tilde{T}(x_0) = x_0$ so $\tilde{T}px = px$ ($p : \mathbb{R}^{2n} - 0 \to \mathbb{R}P^{2n-1}$ projection) $\therefore pTx = px$ so $Tx \sim x$.
$\therefore \exists \lambda \neq 0$, $Tx = \lambda x$ contradicting the fact that T has no eigenvalues.

#7 Map the circle onto itself by going around 1½ times $p(t) = (\cos 2\pi t, \sin 2\pi t)$, $t \in [0, \frac{3}{2}]$. The map p is onto and nullhomotopic ($\therefore$ degree 0). Let $p^{n-1} = S \cdots S p$ the $n-1$ times suspension of p. Then degree $p^{n-1} = 0$

So it is nullhomotopic. Show p^{n-1} is onto using Massey p. 189.

#8 Use Mayer Vietoris with $A_1 = \{(s,t) \mid t > \frac{1}{4}\}$, $A_2 = \{(s,t) \mid t < 3/4\}$

Then A_1 and A_2 are contractable and $A_1 \cap A_2 \cong Q$.